ON SOME POINTS IN THE FOUNDATION OF MATHEMATICAL PHYSICS.

THE need that showed itself, in comparatively modern times, of the greatest attainable logical precision in the concepts and methods of science led, in the field of pure mathematics, to the fundamental work of Weierstrass,[1] Cantor, Dedekind, Frege, Peano, and Russell, and, in the field of physics, to those researches of which the most important are associated with the names of Mach and Stallo.[2] In mathematics, the main result has been the proof that all pure mathematics deals exclusively with concepts definable in terms of the fundamental logical concepts, and that all its propositions are deducible from the fundamental logical principles;[3] and, consequently, Kant's view that mathematical reasoning is not strictly formal, but always uses intuitions, that is to say, the *a priori* knowledge of space and time, can be definitely refuted.[4] But, although

[1] In Weierstrass's case, this need can be seen to be in the highest degree *practical,* since many general theorems in the theory of analytic functions, "proved" by the older analysis, show themselves, on examination, to be unsound or insecure, and point the way to considerations of the very foundations of arithmetic. Such a theorem is that on the existence of a point of condensation in an infinite aggregate of real or complex numbers, of which the importance, in the theory of analytic functions, was seen by Briot and Bouquet, who gave a palpably insufficient proof of the theorem in question.

[2] The tendency towards "physical symbolism" (see a note below) is also marked by some of the works of Maxwell, Kirchhoff, Clifford, Hertz, Karl Pearson, Ostwald, and others. (Cf. H. Kleinpeter, *Die Erkenntnistheorie der Naturforschung der Gegenwart,* Leipsic, 1903; H. Höffding, *Moderne Philosophen,* Leipsic, 1905, pp. 98-117).

[3] Cf. Russell, *The Principles of Mathematics,* vol. i, Cambridge, 1903; especially pp. v, 3-9.

[4] Russell, *op. cit.,* pp. 4, 456-461; cf. Couturat, *Les principes des mathématiques,* Paris, 1905, pp. 235-308.

there is thus no such thing as a philosophy of *mathematics,* as distinct from one of logic, in physics the case is different. The above critical discussions have put in a clear light the fact, which is invariably overlooked by the cruder physicists, that the "world" with which we have to do in theoretical mechanics, for example, is but a mathematical scheme whose function it is to imitate, by logical consequences of the properties assigned to it by definition, certain processes of nature as closely as possible.[5] Thus our "dynamical world" may be called a *symbol* of reality, and must not be confused with the reality itself.[6]

[5] This fact forms the basis of Ward's first argument against naturalism in his *Naturalism and Agnosticism,* 2d ed., London, 1903, vol. i.

[6] Ward (*Philosophical Orientation and Scientific Standpoints* [Annual address before the Philosophical Union of the University of California], Berkeley, 1904, p. 8) called those physicists who realize this gap between the concept and the reality "physical symbolists," as distinguished from the "physical realists," who are metaphysicians in spite of themselves.

When, in this address (p. 2), Ward insists on the implication of some reality behind appearance in the very use of the term "phenomenon," there seems to be a misunderstanding of the scientific position. It has, as is well known, often been urged by philosophers (see, for example, Dr. John Caird's *Introduction to the Philosophy of Religion,* new ed., Glasgow, 1901, pp. 14-15; Ward, *Naturalism and Agnosticism,* 2d ed., vol. i, 1903, p. 24, and vol. ii. pp. 275-276) and others (see, for example, H. Spencer, *First Principles,* 6th ed., 1900, p. 13), that, when we say that a thing is only a phenomenon or appearance, we imply that there is something which is not mere appearance but reality. Now, it is not true that to speak of a thing *a* implies that there is something which is not *a* (for example, to speak of an entity does not imply the self-contradiction that there is a non-entity); so that we must conclude that what the philosophers mean is: The word "appearance" is part of an incompleted phrase "appearance of......," and hence an appearance which is not an appearance *of* anything is a contradiction. But it seems to me that men of science just took the word "appearance" or "phenomenon" as a general term for the facts (of consciousness) which Mach has called by the less metaphysical name of "elements" (*Contributions to the Analysis of the Sensations,* English translation, Chicago, 1897, pp. 5, 11, 18), and philosophers are apparently unable to appreciate the procedure of scientific people of taking a word, which may have had a previous meaning, re-defining it, and thereafter giving it no meaning not provided in the definition. It is true that Mach used the word "Erscheinung" in his *Erhaltung der Arbeit* of 1872 without pointing this out explicitly (he explicitly abandons the use of the word "sensation," because it *seemed* to refer to the (hypothetical) ego, in *op. cit.,* p. 18, but it should at least be remembered that the verbal implication in the word "appearance" is a trace of the philosophical point of view of the naive man (cf. Mach, *op. cit.,* pp. 10, 25-26) who thinks that experience is the knowable result of the interaction between an unknowable thing-in-itself and an unknowable ego. When a man of science talks of "phenomena" he no more implies the existence of another reality than did Mr. Vincent Crummles in *Nicholas Nickleby,* when he spoke of "The Infant Phenomenon."

Again, if we were to use the word "phenomenon" literally, with its reference to a crude philosophy, to talk of "phenomena *per se*" would be to talk

The present paper contains suggestions for the application of the more refined mathematical conceptions—such as those of "continuity" and "motion"—to the mathematical determination of our image of reality. When once we have begun to set up, for what is, at bottom, the *practical* need of completing facts in thought,[7] the mathematical image of the universe, we have left behind all the philosophical problems, and we have only to look to the progress of sciences like those of electricity, chemistry, and psychology for the gradual completion of the image, or model, of the universe, and for the consequent precise answering of epistemological questions.[8] And the mathematician is completely master of his model; he can repeat the occur-

of "those phenomena which are not phenomena, but realities." But, if we use the word in the sense of Mach's "elements of consciousness" and avoid (cf. Mach, *op. cit.*, p. 20) the question: *whose* consciousness?, which arises also from the verbally implied reference to a crude philosophy, I see no reason against calling the phenomena the only reality the "outer" world has for us.

[7] Cf. Mach, *op. cit.* pp,. 151 note, 171-176; *Popular Scientific Lectures*, 3d ed., Chicago, 1898, pp. 236-258, especially p. 253; pp. 186-213; *Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt*, 4te Aufl., Leipsic, 1901, pp. 510-528 (This work was translated into English by T. J. McCormack under the title *The Science of Mechanics*, Chicago: Open Court Pub. Co., 1893; 2d ed. 1902); *Die Principien der Wärmelehre historisch-kritisch entwickelt*, 2te Aufl., Leipsic, 1900, pp. 365-366, 391-405.

[8] If we neglect the psychological aspect of the answer to Kant's question: How is nature, as a system of laws, possible?, which was sketched in Ward's address, pp. 11 *et seq.*, and *Naturalism and Agnosticism*, vol. ii, and only concern ourselves with what is implied logically by the existence of the science—our above "model" of nature, we arrive at a series of exact answers, expressed, of course, in mathematical language, to epistemological questions, and this series can only be completed when our model is sufficiently complete. As yet, our model may be judged complete in dynamical respects, at least. The postulate (if postulate it can rightly be called) that a model is possible seems to me to be the "postulate of the comprehensibility (or uniformity) of nature."

In this way, it seems to me that the only remaining function for the philosopher, as distinguished from the logician, is to give the mind such acquaintanceship (which Heymans seems to call "absolute knowledge"; *Einführung in die Metaphysik auf Grundlage der Erfahrung*, Leipsic, 1905, pp. 1-2) with the conceptions of science (including such subjects as æsthetics and morals, which are as yet hardly more than possible sciences) as it has with redness or the taste of a pineapple (cf. Russell, *op. cit.*, p. v, where also the failure of "the search with a mental telescope" for the notion of *class*, was spoken of).

There are some suggestive remarks on epistemology in Stallo, *op. cit.*, pp. xxxv-xlii, 25-26, 68-69; and Mach, in his *Erhaltung der Arbeit*, of 1872, emphasized the "logical root" in the principle of the conservation of energy. But I will reserve a closer examination of these questions for another opportunity.

rences in his universe as often as he likes, he can make his "sun" stand still, or hasten, in order that he may publish the "Nautical Almanac" several years ahead of time. His position contrasts with that of the mere observer like that of the man who had thirty dollars in his mind with that of the man who had thirty dollars in his purse, in Kant's illustration of the untenability in logic of the ontological proof of the existence of God.[9] The mathematical physicist can, without difficulty, become rich beyond the wildest dreams of avarice,—but by dream-gold.

I.

The first stage in the construction of our image is naturally the formation of a *"dynamical* world." This world closely resembles that of Russell,[10] but it seems to me essential that the "space" should be an aggregate of complex *numbers* (with three unities), and the time should be an aggregate of real *numbers,* for only then can we describe the motions in this world by differential equations.[11] If, on the other hand, the space and time were (as Russell assumed)[12] any continua which are defined *purely ordinally,* we could give a meaning to a certain position being a "function" (even a continuous one)[13] of

[9] Cf. also Russell, *Mind,* 1905, p. 491.
It has been maintained with great ability, that this "proof" is not to be regarded as an attempt at a strictly logical proof, but as meaning that our whole conscious life is based on a universal self-consciousness (Caird, *op. cit.,* pp. 144-150.)

[10] *Op. cit.,* pp. 480-481.

[11] Cf. Russell, *op. cit.,* pp. 326-329. There, even the notion of the *continuity* of a function was defined, after Dini, in a manner which is not purely ordinal, but is applicable, in the first instance, only to series of *numbers.* I have shown (see the next note but one) that the notion of continuity of a function can be given a purely ordinal (not necessarily connected with a series of numbers) definition, but not the notion of a differential quotient.

[12] *Op. cit.,* pp. 288, 437-440, 473.

[13] I have shown (*Journ. für Math.,* Bd. CXXVIII, 1905, pp. 182-199.) that not only *function* (as Russell emphasized, *op. cit.,* pp. 263-267), but also *continuous function,* can be conceived in a purely ordinal manner. That a continuous function could be thus conceived was tacitly assumed by Russell (*op. cit.,* p. 480) in speaking of a continuous function whose argument was a series (not necessarily of numbers).

the time, but we could not to a differential quotient like dx/dt. From this it follows that not only do we have absolute time and position in our dynamical world in contradistinction to our real world, as Ward has already pointed out, but that "measurement" which we apply in actual experiments, has only an analogue in mathematics by a convention.[14] It may be remarked, also, that Russell's[15] statement that the world *may*, in spite of Mach's arguments, quoted by Ward, be twice given, only applies to the "dynamical world" (for there a mathematician is a sort of demiurge), and Mach and Ward were only speaking of our real world.[16]

[14] The "distance" of two "points in an arithmetical space," (x, y) and (x', y'), may be defined, in a way recalling our empirical notion of distance, but which is not purely ordinal, as $+\sqrt{(x-x')^2+(y-y')^2}$, or as Jordan's "écart," $|x-x'|+|y-y'|$. Cf. also Russell, *op. cit.*, pp. 425-428.

[15] *Op. cit.*, pp. 492-493.

[16] The attempt to make certain purely mathematical concepts the foundation of reality may be illustrated by the ludicrous attempts at the substantialization of "ether" and "space" (cf. Stallo, *The Concepts and Theories of Modern Physics*, 4th ed. London, 1900, pp. xxiv-xxvi, 43-44, 227-230, 247; Ward, *op. cit.*, vol. i, pp. 128-138.

Helmholtz investigated mathematically the properties of rotational movement in an absolutely homogeneous, incompressible, perfect fluid, and Lord Kelvin based on these researches his well-known speculations on "vortex-atoms." Now, in order to realize the difference between these two investigations, let us reflect that what a mathematician can only mean by such a "fluid" is a certain aggregate of numbers (the numbers are here complex numbers with three independent unities, so as to mimic the space of experience), while the transformations or "motions" of parts of it underlie certain conditions which are described picturesquely by the words "incompressibility" and "perfection." The whole problem is purely mathematical; but, because, in nature, a process may be more or less exactly described by it when these natural processes are referred to certain determinable axes regarded as "fixed" and the various points of natural space are put in a metrical one-one correspondence with one aggregate of numbers, the names of the natural objects or properties have been transferred, usually in an uncritical manner, to the mathematical *images* of them. This it is that has given rise to the many attempts (of which Lord Kelvin's is an example) to attempt, inversely, to make the mathematical image the essence of the reality. Then we at once get the valid objections that the motion in such a fluid (supposed real) is not sensible, and that the "fluid" itself is, like "atoms" (cf. Stallo, *op. cit.*, p. 156 note), "things-in-themselves" (Stallo, *op. cit.*, p. 159; Mach, *op. cit.*, pp. 6, 23 note), or the result of attempts to reify the mathematical conception of "space" (Stallo, *op. cit.*, pp. 214-215)—in a conception which has arisen psychologically, though not logically, from the "space" of the physicists (cf. Mach, *op. cit.*, p. 55; *Die Principien der Wärmelehre*, 1900, pp. 456-457 and 71-77 [on the physical "continuum"]; *Mechanik*, 1901, pp. 232-253; *Erhaltung der Arbeit*, 1872, pp. 56-57), nothing at all.

The widespread notion that geometry is a science dealing with pure

2.

There comes the question of the meaning of such terms as "causality" in our image. Mach[17] has, now, formulated the law of causality in the form: To every phenomenon α belongs a group which uniquely determines it (of which it is a one-valued function). Even this very general formulation (which includes Petzoldt's[18] "Gesetz der Eindeutigkeit") leads to important consequences: it shows the identity of the two forms in which the principle of the conservation of energy has been expressed, and can itself be transformed mathematically into what is an equivalent of Poincaré's[19] rather vague generalization of the principle of conservation of energy: "Il y a quelque chose qui demeure constant."

From this point, then, there opens a field of *mathematical* research. We have seen that the functions occurring in that aspect of the law of causality with which we have to do in mathematical physics are one-valued functions of many real variables ("phenomena" or "elements," as Mach calls them), which, in the case of dynamics, can

space—a notion not to be confused with the space of the physicists or number-aggregates—leads to many contradictions in geometry and mechanics. There is much that can be done in the investigation of the purely ordinal properties of series, but, in order that arithmetical notions, such as that of a differential quotient, may be used, our "space" and "time" must be numerical aggregates. And when this is the case, the difficulties as to absolute or relative position and motion cease to appear; while position and motion in a "pure" space has the same difficulties as in the "ether" just described. An example of an attempt to retain "the philosophical dictum that all motion is relative," a pure space, and differential equations of motion, in one book on mechanics, is given by A. E. H. Love's *Theoretical Mechanics*, (Cambridge, 1897). A "motion relative to a frame of reference" is only satisfactory if we can, so to speak, ear-mark the frame; and there is no way of doing this in a "pure" space.

[17] Cf. *Die Geschichte und die Wurzel des Satzes von der Erhaltung der Arbeit*, Prag, 1872, pp. 35-37; *Mechanik*, 1901, pp.513-515.

[18] J. Petzoldt, "Maxima, Minima und Oekonomie," *Vierteljahrsschrift für wiss. Philosophie*, Jahrg. XIV, 1890, pp. 206, 354, 417; "Das Gesetz der Eindeutigkeit," *ibid.*, Jahrg. XIX, 1895, pp. 146-203; cf. Mach, *Mechanik*, 1901, p. 409.

[19] *La Science et l'hypothèse*, éd. revue et corrigée, Paris, p. 153.

be replaced by *one* real variable, the time; there arises the question as to what restrictions on these functions (to be continuous, analytic,...) are necessary or convenient in that mathematical image of the real world which we call the "dynamical world."[20] I will illustrate this.

Suppose that x is a co-ordinate in a dynamical problem: in any particular problem our object is to find an expression of x in terms of the time (t), and, in conformity with the law of causality, we assume that x is a (one-valued) function of t:

$$x = f(t).$$

Now, in order that the real number called "the velocity of x at $t=t_0$" ($[dx/dt]t=t_0$) should exist, $f(t)$ must be differentiable at $t=t_0$; and, for this, continuity of $f(t)$ is necessary but not sufficient, while an "analytic" character of $f(t)$ near $t=t_0$ is sufficient but not necessary; and so on. We can, indeed, contemplate the most various dynamical worlds; for example, while the law of causality holds, a moving point may have no velocity at any point of its (continuous) path. For this, we have only to suppose $f(t)$ to be a continuous function without a derivative, as Appell and Jannaud have done.[21]

This thorough investigation of the dynamical world with all the resources of the modern theories of functions and of aggregates, although it seems very far removed from what are commonly supposed to be the objects of mathematical physics, appears to me to be the only way in which we can be sure that the image of reality at which we aim,

[20] For this, cf. A. Voss, "Die Prinzipien der rationellen Mechanik," *Encykl. der math. Wiss.*, iv. I., pp. 20-30.

[21] "Remarques sur l'introduction de functions continues n'ayant pas de dérivée dans les éléments de la dynamique," *Compt. rend.*, T. XCIII, 1881, p. 1005; *Archiv für Math.*, Bd. LXVII, 1882, p. 160.

Similar examples are afforded by the contemplation of one of Peano's "curves" which fill a square, as the path of a moving point (cf. A. Schönflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1900, pp. 121-125; W. H. Young and [Mrs.] Grace Chisholm Young, *The Theory of Sets of Points*, Cambridge, 1906, pp. 165-170, 219-232, 291-292.)

by successive approximations, is what Hertz[22] called a "logically *permissible* image."

In this way, we shall come across such problems as the following: The law of the conservation of energy is now recognized as a specialized form (the form of which has been discovered by observation) of the law of causality; what limitations on the form of the functions involved in the latter law does this specialization involve?

3

Every theorem in the theory of functions or in that of differential equations brings to light a property of the dynamical world which sometimes appears very surprising. Thus, the (not yet completely solved) problem of finding the necessary and sufficient conditions under which a solution of a system of ordinary (real) differential equations can exist, would give us the necessary and sufficient conditions that a moving body which is somewhere at some time should be somewhere at some other time. But this surprising appearance is only due to our use of phrases we use about real things to describe occurrences in the dynamical world,—a world which has no secrets from us, if our minds are powerful enough, but which is only an image of the real world, although an image which, perhaps, may become indefinitely like the original in certain respects.

[22] Heinrich Hertz, *The Principles of Mechanics Presented in a New Form,* translated from his *Werke,* Bd. iii, by D. E. Jones and J. T. Walley, London, 1899, p. 2.

In this place, it seems proper to refer to Cantor's discovery, at the end of his third paper "Ueber unendliche, lineare Punktmannichfaltigkeiten" (*Math. Annalen.,* Bd., XX, 1882, pp., 113-121), that a continuous motion may be possible in a discontinuous space; and his consequent suggestion that an attempt might be made to form, for purposes of comparison, a "modified mechanics" for spaces of the kind to be described. From a continuous arithmetical space of two or more dimensions, remove an enumerable but everywhere-dense aggregate (such as that of all rational, or of all algebraic numbers). Any two points of the remainder (A) can always be connected by a *continuous* line (formed of a succession of arcs of circles), all the points of which belong to A.

4.

This investigation of the principles of mathematical physics permits us to clear up some difficulties which every mathematician accustomed to exactness feels in current presentations of mathematical physics. For example, in investigating the vibrations of a stretched cord after the method introduced by Lagrange,[23]—in which we contemplate the vibrations of a finite number (n) of masses placed at equal distances along an elastic mass-less "string," and then go to the limit by supposing n to increase *ad infinitum*,—any one who is acquainted with the theory of aggregates makes the following remark. If the cord is a continuum (of type θ); since we cannot say more of the limiting case above than that the mass-points be everywhere dense on the string (a condition which need only make these mass-points of type η) a passage to the limit does *not* give us the cord, since a series of type θ always contains one of type η, but a series of type η never contains one of type θ.

This difficulty, now, vanishes if we assume, as is natural if we wish to make our image conform to nature, that the cord always represents a *continuous* function of its position at rest, a straight line (i. e., "the cord is never to break"). For we know that a continuous function is determined for a continuum (of type θ) when it is given for merely an everywhere-dense aggregate (of type η).

5.

Finally, it may be mentioned that if, and only if, the functions occurring in our formulation of the law of causality, are *rational* functions, a *finite* number of particular

[23] "Recherches sur la nature et sur la propagation du son," *Misc. Taur.*, t. i., 1759, and "Nouvelles recherches," *ibid.*, t. ii., 1760-1761; *Œuvres*, t. i.; cf. Mach, *Wärmelehre*, pp. 110-111; R. Reiff, *Geschichte der unendlichen Reihen*, Tübingen, 1889, pp. 132-134.

determinations of the values of the function suffice to construct the whole function. In other words, the problem of interpolation is here, and here only, determinate. The translation of this into a form suitable for a simple mechanical case is: If the coordinates $q_1, q_2, \ldots, q_\nu$, of a mechanical system are all rational and whole functions of t of degrees which do not exceed n, the finding of the values of the q's for any $n+1$ particular values of t alone suffices for the determination of the complete expression of the q's as functions of t.

The practical importance of this results from the known fact that *any* real, one-valued, continuous function of t can be approximated *quam proxime* by a rational and whole function of t.[24] It is, then, a plausible supposition that the postulate that all the functions occurring in the mathematical formulation of the law of causality are rational and whole functions suffice for the construction of a "dynamical world" which copies, with an approximateness of which any discrepancy is not observable, the scientific aspect of the real world.

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[24] This theorem was discovered by Weierstrass ("Ueber die analytische Darstellbarkeit sogenannter willkürlicher Funktionen reeller Argumente," 1885, *Werke*, Bd. iii, pp. 1-37), and other proofs have been given by Picard, Volterra, Runge, Lebesgue, and Mittag-Leffler (cf. Borel, *Leçons sur les fonctions de variables réelles et les développements en séries de polynomes*, Paris, 1905, pp. 50-92).

The extension of the problem of interpolation from *rational* functions to analytic functions in general, and some allied questions was treated in my above-cited paper in the *Journal für Mathematik* for 1905.